Quantum telecloning of optical images: Multiuser parallel quantum channel

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We introduce and describe in detail a different quantum-information protocol: quantum telecloning of optical images or spatially multimode states of light. This protocol is a natural generalization of a continuous-variable quantum telecloning of single-mode states, investigated earlier theoretically and recently realized experimentally. Our protocol is an example of a multiuser parallel quantum channel. In order to realize this scheme one needs a multimode multiparticle entanglement between the original and the target systems. We calculate the fidelity of quantum parallel telecloning in dependence on the spatiotemporal scales of our scheme and on the number of pixels. The fidelity is superior to the best classical fidelity that can be achieved without this kind of entanglement.

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I. INTRODUCTION

Quantum-information processing has offered several protocols which do not have classical analogs. Quantum cloning is an example of such protocols. While the no-cloning theorem [1] forbids perfect duplication of an arbitrary quantum state, it was shown that there is always a possibility of imperfect cloning for both discrete variables, or qubits [2–4], and continuous variables [5–7].

Another well-known quantum-information protocol is quantum teleportation. It allows for perfect transfer of an arbitrary quantum state from one place to another using a classical communication channel and quantum entanglement shared by two parties. This protocol was proposed initially for discrete variables [8] and later for continuous variables [9–11]. Experimental realizations of quantum teleportation were demonstrated for single-photon polarization states [12] and single-mode coherent states [13–15].

Recent theoretical works [16–18] have generalized the single-spatial-mode teleportation scheme to a spatially multimode case thus allowing for teleportation of parallel optical signals or optical images. Teleportation of optical images is an example of a parallel quantum channel. The fidelity of classical teleportation of an N-mode parallel channel is equal to $F_{ct}=(1/2)^N$. Therefore for a large number of modes $N$ this fidelity becomes very small. On the contrary, it was demonstrated in [16,17] that the fidelity of quantum parallel teleportation using spatial-multimode entangled states in an optimal case scales as $F_{ct}=[1+\exp(-2r)]^{-N}$, where $r$ is the squeezing parameter. Under optimal conditions found in [16,17] this fidelity can be made close to unity even for $N$-mode parallel teleportation.

Unification of quantum cloning with quantum teleportation provides a different quantum information protocol: quantum teleporting or cloning at a distance. This protocol allows for distribution of quantum information simultaneously to several receivers and is an example of a “multiuser quantum channel” or quantum network. Of course, due to the no-cloning theorem, the fidelity of quantum teleporting can never reach unity. However, the fidelity of teleporting based on quantum entanglement is always higher than the fidelity of an equivalent classical protocol without the use of entanglement. Theoretical proposals of teleportation can be found in Ref. [19] for qubits and in Ref. [20] for continuous variables, and experimental realizations for qubits in [21] and for continuous variables in [22].

In this paper we generalize the continuous-variable quantum teleporting protocol from single-mode quantum states to spatially multimode states. This generalized scheme gives a possibility for quantum teleporting in parallel of stationary or time-dependent optical images. Our protocol can be also considered as a “multiuser parallel quantum channel” consisting of $N$ modes. We will show in particular that the fidelity of optimum “quantum telecloning” in such a channel is given by $F=(2/3)^N$, while the best fidelity achieved by “classical telecloning” is equal to $F_{ct}=(1/2)^N$. Let us stress here that this difference between the quantum and classical telecloning appears even for the input multimode coherent states considered in this paper. While with growing $N$ both fidelities decrease to zero, the difference between classical and quantum protocols becomes more pronounced for a parallel channel with large number of modes $N$. In this paper we consider as an example the $1\rightarrow 2$ parallel telecloning. However, our results can be easily generalized to a $M\rightarrow M'$ telecloning scheme. Our telecloning protocol can be performed both with and without frequency conversion, i.e., the output images can have the same or different carrier frequency compared to the original image. For the sake of generality we shall consider here the parallel telecloning scheme with frequency conversion.

The key feature of the multimode quantum telecloning protocol is the multimode multiparticle entanglement shared by the original and the target systems. In our case the source of spatially multimode entanglement between the input image at frequency $\omega_1$ and two cloned images at frequency $\omega_2$ is a type-I traveling-wave nondegenerate optical parametric amplifier (OPA). We will demonstrate that for the optimal telecloning of optical images one needs only a finite degree of squeezing. Earlier this conclusion was obtained for single-mode telecloning protocol [20,22]. This result is very encouraging compared to the quantum teleportation where in order to obtain the maximum fidelity one needs perfect squeezing [16].
In this section we shall give a detailed description of the protocol for telecloning of optical images. The optical scheme of telecloning is shown in Fig. 1. We consider the scheme with frequency conversion, when the electromagnetic wave carrying an input image has the carrier frequency $\omega_1$ and the two output waves with telecloned images have the frequencies $\omega_2$. An input image which is to be telecloned from Alice to Bob and Claire, is described by a slowly varying field operator $\hat{A}(\vec{p}, t)$, where $\vec{p} = (x, y)$ is a two-dimensional transverse coordinate. The input wave is divided into two secondary waves $B_1$ and $B_2$ by a 50:50 beam splitter $BS_1$, and two quadrature components of these secondary waves are homodyne detected using two local oscillators $LO_1$ and $LO_2$, two 50:50 beam splitters $BS_2$ and $BS_3$, and four efficient charge coupled device (CCD) matrices with appropriate spatial resolution.

The local oscillator waves $LO_1$ and $LO_2$ have the same frequency $\omega_1$, as the carrier frequency of the input wave. The photocurrent densities from the CCD matrices, containing information about the spatiotemporal quantum fluctuations of the input image, are transmitted from Alice to Bob and Claire and are used for creation of two output clones $\hat{A}_1(\vec{p}, t)$, $\hat{A}_2(\vec{p}, t)$, $n=1,2$. Two pairs of multichannel modulators $m_{1x}$, $m_{1y}$, for Bob and $m_{2x}$, $m_{2y}$, for Claire, perform spatiotemporal modulation of two coherent plane waves with carrier frequency $\omega_2$.

As stressed in Refs. [20,22], single-mode quantum $1 \rightarrow 2$ telecloning is based on tripartite continuous variable entanglement between Alice, Bob, and Claire. The same remains true for the multimode $1 \rightarrow 2$ telecloning; in order to obtain parallel multimode telecloning we need multimode tripartite entanglement between these three parties. The key ingredient for creation of this entanglement is a pair of spatially multimode Einstein-Podolsky-Rosen (EPR) fields $\hat{S}_1(\vec{p}, t)$ with frequency $\omega_1$ and $\hat{S}_2(\vec{p}, t)$ with frequency $\omega_2$. Since entanglement between different carrier frequencies is needed, these EPR fields are created by a type-I traveling-wave nondegenerate OPA. The dichroic mirror $M$ reflects the wave $\hat{S}_1(\vec{p}, t)$ and transmits the wave $\hat{S}_2(\vec{p}, t)$. In the case of perfect reflectivity there are no frequency-matched vacuum fluctuations entering from the open ports of $M$ into the corresponding secondary waves. The second EPR wave $\hat{S}_2(\vec{p}, t)$ is further divided into two waves $\hat{C}_1(\vec{p}, t)$ and $\hat{C}_2(\vec{p}, t)$ by a 50:50 beam splitter $BS_4$. The vacuum fluctuations entering into the scheme from the open port of the beam splitter $BS_4$ are described by the field operator $\hat{C}_{vac}(\vec{p}, t)$.

Let us introduce the slowly varying spatiotemporal annihilation and creation operators $\hat{S}_\mu(\vec{p}, t)$ and $\hat{S}_\mu^+(\vec{p}, t)$, $\mu=1,2$ of the electromagnetic waves with central frequencies $\omega_1$ and $\omega_2$ at the output of the nondegenerate OPA. The frequencies $\omega_1$ and $\omega_2$ obey the condition of energy conservation, $\omega_1 + \omega_2 = \omega_\mu$, where $\omega_\mu$ is the frequency of the pump wave. The field operators are normalized so that $(\hat{S}_\mu(\vec{p}, t)\hat{S}_\mu^+(\vec{p}, t))$ gives the mean value of the irradiance, expressed in photons/cm$^2$/s.

The transformation of the input fields $\hat{E}_\mu(\vec{p}, t)$ of the nondegenerate OPA in the vacuum state into the output fields $\hat{S}_\mu(\vec{p}, t)$ in the broadband multimode squeezed state is described in terms of the Fourier components of these operators in frequency and spatial-frequency domain. $\hat{S}(\vec{p}, t) \rightarrow \hat{s}(\vec{q}, \Omega)$. The squeezing transformation performed by a
nondegenerate OPA can be written as follows:
\[
\hat{\sigma}_\mu(q, \Omega) = U_\mu(q, \Omega)\hat{\sigma}_\mu(q, \Omega) + V_\mu(q, \Omega)\hat{\sigma}_\mu(-q, -\Omega),
\]
where the coefficients \( U_\mu(q, \Omega) \) and \( V_\mu(q, \Omega) \) depend on the amplitude of the pump field, nonlinear susceptibility, and the phase-matching condition. The explicit form of \( U_\mu(q, \Omega) \) and \( V_\mu(q, \Omega) \) is discussed in the Appendix. The EPR correlations between the fields \( \hat{S}_1(\rho, t) \) and \( \hat{S}_2(\rho, t) \) are determined by two parameters, namely, the orientation angle \( \psi_\mu(q, \Omega) \) of the major axis of the squeezing ellipse [23],
\[
\psi_\mu(q, \Omega) = \frac{1}{2} \arg[U_\mu(q, \Omega)V_{\mu'}(-q, -\Omega)],
\]
and the degree of squeezing \( r_{\mu}(q, \Omega) \),
\[
\exp[\pm r_{\mu}(q, \Omega)] = |U_{\mu'}(q, \Omega)| \pm |V_{\mu'}(q, \Omega)|.
\]
It should be noted that in Fig. 1 we have chosen the input field at the frequency \( \omega_1 \) and the output fields both at the frequencies \( \omega_2 \), that correspond to \( \mu = 1 \) and \( \mu' = 2 \) in Eqs. (2.2) and (2.3), but with the same source of entangled beams the telecloning with frequency conversion can be performed from \( \omega_2 \) to \( \omega_1 \) as well. In what follows we shall consider the telecloning from \( \omega_1 \) to \( \omega_2 \).

Let us consider the evolution of the fields in the scheme. The input field \( \hat{A}(\rho, t) \) is mixed with one EPR beam \( \hat{S}_1(\rho, t) \) at the 50:50 beam splitter BS1. The secondary waves \( \hat{B}_x \) and \( \hat{B}_y \),
\[
\hat{B}_{x,y}(\rho, t) = \frac{1}{\sqrt{2}}[\hat{\sigma}_2(\rho, t) \pm \hat{S}_2(\rho, t)],
\]
with the \(+(-)\) sign corresponding to the \( x(y) \) component, are detected by means of two balanced homodyne detectors with two local oscillators LOx and LOy and four efficient CCD photomultipliers. We shall assume that the local oscillators LOx and LOy have wave fronts and the complex amplitudes \( B_0 \) and \( iB_0 \), respectively, where \( B_0 \) is taken real. The different photocurrent densities \( I_x(\rho, t) \) and \( I_y(\rho, t) \) collected from individual pixels of these matrices carry the information about the spatiotemporal quantum fluctuations of the quadrature components of \( \hat{B}_x(\rho, t) \) and \( \hat{B}_y(\rho, t) \), phase matched with LOx and LOy:
\[
I_x(\rho, t) = B_0^2 [B_x(\rho, t) + B_y(\rho, t)],
\]
\[
I_y(\rho, t) = B_0^2 [B_x(\rho, t) - B_y(\rho, t)].
\]
The photocurrent densities \( I_x(\rho, t) \) and \( I_y(\rho, t) \) are sent from Alice to Bob and Claire via two multichannel classical communication lines. These photocurrent densities are used for the spatiotemporal modulation of two external coherent light waves of frequency \( \omega_2 \), shifted by phase by \( \pi/2 \), by means of two pairs of multichannel modulators \( m_{1x}, m_{1y} \) for Bob and \( m_{2x}, m_{2y} \) for Claire.

The telecloned fields \( \hat{A}_n(\rho, t), n=1,2 \) are created by mixing of these modulated waves with the corresponding EPR waves \( \hat{C}_n(\rho, t) \) for Bob and \( \hat{C}_n(\rho, t) \) for Claire at the mirrors \( M_1 \) and \( M_2 \) with almost perfect reflectivity for the EPR beams. By choosing appropriately the modulation coefficients and the mirror transmissions in order to obtain the optimal performance, we find the telecloned fields \( \hat{A}_n(\rho, t) \),
\[
\hat{A}_n(\rho, t) = \hat{A}(\rho, t) + \hat{F}_n(\rho, t),
\]
where
\[
\hat{F}_n(\rho, t) = \pm \frac{1}{\sqrt{2}} \hat{C}_n(\rho, t) + \frac{1}{\sqrt{2}} \hat{S}_2(\rho, t) + \hat{S}_1(\rho, t),
\]
are the operators describing the noise added to the telecloned fields. The index \( 1(2) \) in \( \hat{F}_n(\rho, t) \) corresponds to the \(+(-)\) in the term \( \pm \frac{1}{\sqrt{2}} \hat{C}_n(\rho, t) \). It can be demonstrated that the added noise satisfies the commutation relations
\[
[\hat{F}_n(\rho, t), \hat{F}_n(\rho', t')^\dagger] = [\hat{F}_n(\rho', t), \hat{F}_n(\rho, t')^\dagger] = 0,
\]
and therefore can be considered as classical.

III. QUANTUM FLUCTUATIONS OF THE TELECLONED FIELDS

In this section we shall analyze the quantum statistics of the noise field added in the telecloning process. In particular, we shall calculate the covariance matrix of the quadrature components of the added noise field and demonstrate that the added noise is Gaussian. These results will be used in the next section for calculation of the reduced fidelity for the multimode telecloning.

For the noise field (2.7) in the Fourier domain we obtain
\[
\hat{f}_n(q, \Omega) = \pm \frac{1}{\sqrt{2}} \hat{C}_n(q, \Omega) + \hat{\xi}_2(q, \Omega)\hat{\xi}_2^\dagger(-q, -\Omega),
\]
where \( \hat{\xi}_2(q, \Omega) \) and \( \hat{\xi}_1(q, -\Omega) \) are
\[
\hat{\xi}_2(q, \Omega) = \frac{1}{\sqrt{2}} U_2(q, \Omega) + V_1^\dagger(-q, -\Omega),
\]
\[
\hat{\xi}_1^\dagger(-q, -\Omega) = U_1(-q, -\Omega) + \frac{1}{\sqrt{2}} V_2(q, \Omega),
\]
and \( \hat{\xi}_2(q, \Omega) \) and \( \hat{\xi}_2(q, \Omega) \) are the operator-valued Fourier amplitudes of the photon annihilation operators in the vacuum state at the input of the OPA [see Eq. (2.1)]. The correlation functions of the noise amplitudes \( \hat{f}_n(q, \Omega) \) and \( \hat{f}_n^\dagger(q, \Omega) \) are easily found as
\[
\langle \hat{f}_n(q, \Omega)\hat{f}_n^\dagger(q', \Omega') \rangle = (2\pi)^3 \left[ \frac{1}{2} + |\hat{\xi}_2(q, \Omega)|^2 \right] \delta(q - q') \delta(\Omega - \Omega').
\]
In the spatiotemporal domain the noise fields $F_n(\hat{\rho}, t)$ are given by

$$\hat{F}_n(\hat{\rho}, t) = \frac{1}{\sqrt{2}} \hat{C}_{\text{ev}}(\hat{\rho}, t) + \frac{1}{(2\pi)^2} \int d\tilde{\rho} dt_0 (\hat{\xi}_{\tilde{\rho}}(\hat{\rho} - \tilde{\rho}, t - t_0)$$

$$\times \hat{E}_{\tilde{\rho}}(\tilde{\rho}, t_0) + \hat{\xi}_0^{\dagger}(\hat{\rho} - \tilde{\rho}, t - t_0) \hat{E}_{\tilde{\rho}}(\tilde{\rho}, t_0)),$$

and the second-order correlation functions of the noise fields are found in the form

$$\langle \hat{F}_n(\hat{\rho}, t) \hat{F}^\dagger_m(\hat{\rho}', t') \rangle = G_m(\hat{\rho} - \hat{\rho}', t - t'),$$

$$\langle \hat{F}_n(\hat{\rho}, t) \hat{F}_m(\hat{\rho}', t') \rangle = 0. \quad (3.5)$$

Here $G_n(\hat{\rho}, t)$ with $n=1, 2$ are the Green functions corresponding to the individual clones. In our case the two clones are symmetrical and $G_1(\hat{\rho}, t) = G_2(\hat{\rho}, t) = G(\hat{\rho}, t)$.

By using Eqs. (2.2) and (2.3), we find the Fourier transform of the Green function $G(\hat{\rho}, t)$ in the form

$$G(\hat{q}, \Omega) = \frac{1}{2\pi} \frac{1}{2} \left( \frac{\sqrt{2} + 1}{2} \exp(-2\sqrt{2} - 1) \right)^2 \left( \frac{\sqrt{2} - 1}{2} \exp(-2\sqrt{2} + 1) \right)^2 \times$$

$$\times \cos^2 \theta^p(\hat{q}, \Omega) + \left( \frac{\sqrt{2} + 1}{2} \exp(-2\sqrt{2} + 1) \right)^2 \sin^2 \theta^p(\hat{q}, \Omega). \quad (3.6)$$

Let us note that when squeezing and entanglement are not present, $r(\hat{q}, \Omega) = 0$, we have $G(\hat{q}, \Omega) = 1$, and the Green function $G(\hat{\rho}, t)$ is $\delta$-correlated in space and time,

$$G(\hat{\rho}, t) = \delta(\hat{\rho}) \delta(t). \quad (3.7)$$

This case corresponds to the best fidelity of classical telecloning.

The absolute minimum of $G(\hat{q}, \Omega)$ in Eq. (3.6) is reached for the following values of $\theta^p(\hat{q}, \Omega)$ and $r^p(\hat{q}, \Omega)$:

$$\theta^p(\hat{q}, \Omega) = \frac{\pi}{2}, \quad \exp[r^p(\hat{q}, \Omega)] = \sqrt{\frac{\sqrt{2} + 1}{\sqrt{2} - 1}} = 2.4,$$

or approximately 7.7 dB of squeezing. This result was obtained earlier in Ref. [20] for single-mode telecloning. In this optimal case we have $G(\hat{q}, \Omega) = 1/2$ and the optimal fidelity of quantum telecloning for the input coherent states.

The statistics of the noise fields $\hat{F}_n(\hat{\rho}, t)$ from Eq. (2.7) are determined in the most general form by the characteristic functional,

$$\chi_n([\lambda(\hat{\rho}, t)], [\lambda^*(\hat{\rho}, t)]) = \langle \text{in} | \exp \left( \int d\hat{\rho} dt \chi(\hat{\rho}, t) \hat{F}_n(\hat{\rho}, t) \right.$$  

$$- \lambda^*(\hat{\rho}, t) \hat{F}_n(\hat{\rho}, t) \left. \right) | \text{in} \rangle. \quad (3.9)$$

This functional can be calculated similarly to Ref. [17] using the standard techniques of factoring the exponentials of operators:

$$\chi_n([\lambda(\hat{\rho}, t)], [\lambda^*(\hat{\rho}, t)]) = \exp \left( - \int d\hat{\rho} d\hat{\rho}' dt dt' \chi(\hat{\rho}, t) \chi^*(\hat{\rho}', t') \right.$$

$$\times G_n(\hat{\rho} - \hat{\rho}', t - t'). \quad (3.10)$$

From this result we conclude that the noise is Gaussian and, therefore, is completely described by the second-order correlation functions (3.5). An arbitrary correlation function of the noise fields $\hat{F}_n(\hat{\rho}, t)$ and $\hat{F}_n^\dagger(\hat{\rho}, t)$ can be calculated by functional differentiation of $\chi_n([\lambda(\hat{\rho}, t)], [\lambda^*(\hat{\rho}, t)])$ with respect to $\lambda(\hat{\rho}, t)$ and $\lambda^*(\hat{\rho}, t)$. Let us remember that these noise fields commute and, therefore, their ordering is irrelevant.

Due to the frequency-broadband and spatially multimode nature of entanglement in our scheme, the telecloning process will be not instantaneous in time and local in space but “on average” within some spatial area and some finite time interval. For quantitative characterization of the telecloning fidelity we shall use the coarse-grained description, proposed for the multimode teleportation in [16–18]. In order to introduce the coarse-grained field operators, we shall consider averaging of the local output fields $\hat{A}_n(\hat{\rho}, t)$ over a pixel $S_j$ of area $S = A^2$ and over the time window $T_i$ of duration $T$:

$$\hat{A}_n(j, i) = \frac{1}{\sqrt{ST}} \int_{T_i} \int_{S_j} d\hat{\rho} dt \hat{A}_n(\hat{\rho}, t). \quad (3.11)$$

Similar coarse-graining is introduced for the input fields. It is easy to verify that the coarse-grained output field operators satisfy the standard commutation relations for the discrete modes,

$$\hat{X}_n(j, i), \hat{A}_n(j', i') = \delta_{j,j'} \delta_{i,i'}, \quad (3.12)$$

and therefore can be considered as independent degrees of freedom. Next, we shall introduce the quadrature components for these averaged output modes as

$$\hat{X}_n(j, i) = \hat{A}_n(j, i) + \hat{A}_n^\dagger(j, i), \quad \hat{Y}_n(j, i) = -i[\hat{A}_n(j, i) - \hat{A}_n^\dagger(j, i)]. \quad (3.13)$$

Similar definitions will be used for the input and the noise fields.

Using the expression for the output field (2.6), we can rewrite the quadrature components (3.13) via the input field and the noise added in the telecloning process

$$\hat{X}_n(j, i) = \hat{X}(j, i) + \hat{X}_n(j, i), \quad \hat{Y}_n(j, i) = \hat{Y}(j, i) + \hat{Y}_n(j, i), \quad (3.14)$$

where

$$\hat{X}_n(j, i) = \frac{1}{\sqrt{ST}} \int_{S_j} d\hat{\rho} \int_{T_i} dt [\hat{F}_n(\hat{\rho}, t) + \hat{F}_n^\dagger(\hat{\rho}, t)],$$
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\[ \hat{\gamma}_n(j,i) = \frac{-i}{\sqrt{ST}} \int_{S_j} d\tilde{\rho} \int_{T_i} dt \left[ \hat{F}_n(\tilde{\rho},t) - \hat{\gamma}_n(\tilde{\rho},t) \right]. \]  

(3.15)

These operators are linear combinations of Gaussian stochastic variables, independent of the input. Hence \([x_n(j,i)],[y_n(j,i)]\) represent a set of classical Gaussian stochastic variables. Here \(j=1,\ldots,N, i=1,\ldots,K\) is a finite set of indices labeling the pixels and the time intervals of interest. These variables are independent of the input field and have zero mean values. Their statistical properties are completely described in terms of a covariance matrix

\[ C_n(j,j';i,i') = \langle x_n(j,i)x_n(j',i') \rangle = \langle y_n(j,i)y_n(j',i') \rangle. \]

(3.16)

By using Eqs. (3.3), (3.11), and (3.15), we find for the covariance matrix elements

\[ C_n(j,j';i,i') = \frac{1}{ST} \int_{S_j} d\tilde{\rho} d\tilde{\rho}' \int_{T_i} dt dt' \int d\tilde{\rho} d\tilde{\rho}' \]

\[ \times \int d\Omega d\Omega' \left\{ e^{i \tilde{\rho} \cdot \tilde{\rho}' - (\Omega - \Omega') dt t'} \right\} \]

\[ \times \left\{ (f_n(\tilde{\rho},\Omega)f_n'(\tilde{\rho}',\Omega')) + e^{i \tilde{\rho} \cdot \tilde{\rho}' - (\Omega - \Omega') dt t'} \right\} \]

\[ \times \left\{ (f_n(\tilde{\rho},\Omega)f_n'(\tilde{\rho}',\Omega')) \right\} \]

\[ = \frac{1}{ST} \int_{S_j} d\tilde{\rho} d\tilde{\rho}' \int_{T_i} dt dt' \int d\tilde{\rho} d\tilde{\rho}' \]

\[ \times \int d\tilde{\rho} d\tilde{\rho}' \left\{ e^{i \tilde{\rho} \cdot \tilde{\rho}' - (\Omega - \Omega') dt t'} \right\} + c.c. \}

\[ G_n(\tilde{\rho},\Omega). \]

Here \(d\tilde{\rho}=dxdy\), the integration over \(x\) and \(y\) is performed within the intervals \(x \in (x_j-\Delta/2,x_j+\Delta/2), y \in (y_j-\Delta/2,y_j+\Delta/2)\), where \(x_j\) and \(y_j\) are the coordinates of the pixel center, and \(\Delta=\sqrt{S}\) is the pixel size. Similarly, \(t \in (t_i-T/2,t_i+T/2)\). Integration over the pixels areas and the sampling intervals gives

\[ \int_{S_j} d\tilde{\rho} d\tilde{\rho}' \int_{T_i} dt dt' \int d\tilde{\rho} d\tilde{\rho}' \left\{ e^{i \tilde{\rho} \cdot \tilde{\rho}' - (\Omega - \Omega') dt t'} \right\} + c.c. \}

\[ = 2 \cos[\tilde{\rho} \cdot \tilde{\rho}' - (t_i - t_i')] \Delta^4 \]

\[ \times \sin^2 \left( \frac{q_T}{2} \right) \sin^2 \left( \frac{q_T}{2} \right) T^2 \sin^2 \left( \frac{\Omega T}{2} \right). \]

(3.17)

Taking into account that \(G_1(\tilde{\rho},t) = G_2(\tilde{\rho},t) = G(\tilde{\rho},t)\) we have also \(C_1(j,j';i,i') = C_2(j,j';i,i') = C(j,j';i,i')\) with

\[ C(j,j';i,i') = 2 \int d\tilde{\rho} d\tilde{\rho} B_2(\tilde{\rho}) B_T(\Omega) \]

\[ \times \cos[\tilde{\rho} \cdot \tilde{\rho}' - (\Omega(t_i - t_i')] \Omega(\tilde{\rho},\Omega), \]

(3.18)

where we have introduced the \(\delta\)-like functions

\[ B_\Delta(\tilde{q}) = \frac{\Delta^2}{T^2} \sin^2 \left( \frac{q_T}{2} \right) \sin^2 \left( \frac{q_T}{2} \right), \]

(3.19)

\[ B_T(\Omega) = \frac{T}{2} \sin^2 \left( \frac{\Omega T}{2} \right). \]

(3.20)

When the pixel size \(\Delta\) and the time window \(T\) are much larger than the characteristic transversal coherence length \(l_c\) and the coherence time \(T_c\) of the OPA, we can approximate \(B_\Delta(\tilde{q})\) and \(B_T(\Omega)\) by the \(\delta\)-functions. In this case in the absence of the EPR correlations, i.e., when \(r(\tilde{q},\Omega)=0\), we obtain the classical limit of telecloning with the covariance matrix \(C_{cl}(j,j';i,i') = 2 \delta_{jj'} \delta_{ii'}\). In this limit two units of vacuum noise are added at each pixel and each time interval in the telecloning process exactly as in the single-mode telecloning, if we associate the individual pixels and the individual time intervals with the system modes.

When the EPR correlations are present and described by the optimum values of the orientation angle and the squeezing parameter from Eq. (3.8), the covariance matrix in the limit of large pixel and long observation time is equal to \(C(j,j';i,i') = \delta_{jj'} \delta_{ii'}\). In this limit of quantum telecloning only one unit of vacuum noise is added at each pixel and each time interval, which corresponds to the fidelity superior to the best classical one. In the next section we shall evaluate the fidelity of quantum parallel telecloning for some simple cases of the input images consisting of just a few pixels.

IV. FAITHFULNESS OF MULTIMODE TELECLONING

The quality of telecloning protocol is usually described by the fidelity parameter \(F\). For single-mode telecloning of pure quantum states fidelity is defined as

\[ F = \left| \langle \Psi_{in} | \Psi_{out} \rangle \right|^2, \]

(4.1)

where \(|\Psi_{in}\rangle\) and \(|\Psi_{out}\rangle\) are the input and the output quantum states. In the case of \(1 \rightarrow 2\) telecloning we have two output states and should, in principle, consider two corresponding fidelities. However, in our case due to the symmetry of the scheme both fidelities are equal.

As in the case of multimode teleportation [17], in order to describe the fidelity of telecloning for multimode quantum states, we have to identify the relevant set of degrees of freedom, or modes, and to introduce the reduced fidelity for this set. In terms of the coarse-grained description, considered in the previous section, the independent degrees of freedom of the input and the output fields are the discrete field variables \(\hat{A}(j,i)\) and \(\hat{A}_r(j,i)\) corresponding to the pixels \(S_j\) with \(j=1,\ldots,N\) and time windows \(T_i\) with \(i=1,\ldots,K\). If these degrees of freedom are independent at the input and the telecloning process does not add any correlations between different modes (we shall specify below the conditions), the fidelity of the multimode telecloning is equal to the product of the individual fidelities \(F_{ji}\) corresponding to the \(j\)th pixel \(S_j\) and the \(i\)th time interval \(T_i\).
It follows from Eq. (4.2) that in the most simple case when all the individual fidelities are equal, $F_{ij}=F_1$ (unity index standing for one pixel and one time interval), the total fidelity of multimode telecloning is equal to

$$F = (F_1)^{NK}. \quad (4.3)$$

Independent or uncorrelated spatiotemporal modes $S_j$ and $T_j$ can be realized when the size of the pixel $\Delta$ and the time window $T$ are much larger than the corresponding coherence length $\lambda_c$ and the coherence time $T_c$ of the OPA, $\Delta \gg \lambda_c$ and $T \gg T_c$. Without the EPR correlations the corresponding single-mode fidelity is equal to $F_1=1/2$ (classical limit), and the total fidelity is given by $F_1=(1/2)^{NK}$. With the optimum EPR correlations, the corresponding single-mode fidelity is equal to $F_1=2/3$ (quantum limit), and the total fidelity is $F=(2/3)^{NK}$.

In order to evaluate the fidelity parameter $F$ of the multimode telecloning in the general case, we can use the Wigner function of the output quadrature components for the discretized modes corresponding to the pixels $S_j$ and time intervals $T_j$. The calculations are analogous to those in Ref. [17] and we refer to this paper for further details. The resulting $F$ can be written as

$$F = \frac{1}{\det \left[ V^X(j,j';i,i') + \frac{1}{2} \mathcal{C}(j,j';i,i') \right]^{1/2}} \times \frac{1}{\det \left[ V^Y(j,j';i,i') + \frac{1}{2} \mathcal{C}(j,j';i,i') \right]^{1/2}}, \quad (4.4)$$

where

$$V^X(j,j';i,i') = \langle \Delta X(j,i) \Delta X(j',i') \rangle,$$

$$V^Y(j,j';i,i') = \langle \Delta Y(j,i) \Delta Y(j',i') \rangle \quad (4.5)$$

are the elements of the covariance matrix of quantum fluctuations $\Delta X(i,j)$, $\Delta Y(i,j)$ of input Gaussian field, such that $\langle \Delta X(j,i) \Delta Y(j',i') \rangle = 0$. Taking into account that for an input multimode coherent state the input covariance matrix is $V^X(j,j';i,i') = V^Y(j,j';i,i') = \delta_{jj'} \delta_{ii'}$, we can rewrite Eq. (4.4) as

$$F = \frac{1}{\det \left[ \delta_{jj'} \delta_{ii'} + \frac{1}{2} \mathcal{C}(j,j';i,i') \right]}. \quad (4.6)$$

In Fig. 2 we show the numerical results for the reduced fidelity as a function of the relative pixel size $D/l_c$ for three different pixel sets $F_1$, $F_2$, and $F_4$ given on the top of the figure, and different time intervals $T=0.1 T_c$, $T_c$, $10 T_c$ for the optimal squeezing $\exp(r(0,0))=2.4$ from Eq. (3.8).

The coherence time $T_c$ for a frequency nondegenerate OPA is typically estimated as the time delay at the crystal length $l$ between two wave packets centered at the frequen-

FIG. 2. Fidelity of multimode telecloning for three different pixel sets $F_1$, $F_2$, and $F_4$ shown on top of the figure as a function of the relative pixel size $D/l_c$ for three different observation times $T=0.1 T_c$, $T_c$, and $10 T_c$. The squeezing parameter $\exp(r(0,0))=2.4$ is given by Eq. (3.8).

cies $\omega_1$ and $\omega_2$, $T_c \sim |1/v_1 - 1/v_2|$, arising due to the difference of the group velocities $v_n$. Since a typical frequency spectrum of parametric down-conversion is fairly broad, a wave-packet spread due to the group velocity dispersion can also have an effect on squeezing and entanglement, and as a result on the fidelity of the multimode telecloning.

Since the frequency dependence of the squeezing orientation angle arising in a parametric crystal is a pure phase effect, it can be compensated by propagation in a linear medium with a properly chosen frequency dependence of the refraction index. It is worth noting that in many interference experiments with twin photons, a need for similar compensation has been realized some time ago. Such a compensation can be applied to the schemes of multimode teleportation and telecloning and will be discussed in detail in the forthcoming publication [24]. Similarly, the spatial frequency dispersion of the squeezing orientation angle, arising due to diffraction in the OPA, can be compensated by a properly inserted thin lens [17].

The bold lines in Fig. 2 correspond to the case of the phase correction of the squeezing orientation angle by means of a thin lens and a linear dispersion medium. The thin lines are without correction. One can see the superior values of fidelity with the phase correction. The horizontal line at $F=0.87$ corresponds to the maximum fidelity for one pixel of the corresponding multimode teleportation protocol with frequency conversion [18] obtained for the same squeezing parameter. One can see that the fidelity of the multimode telecloning remains inferior to that for the multimode teleportation. This is due to the additional vacuum fluctuations entering into the telecloning scheme from the “open port” at the beam splitter BS$_4$ (see Fig. 1).

Our numerical simulations correspond to one temporal mode, $K=1$. Therefore, according to Eq. (4.3), the asymptotic behavior of three fidelities $F_1$, $F_2$, and $F_4$ for $D/l_c \gg 1$ and $T/T_c \gg 1$ is as follows: $F_1 \rightarrow 2/3=0.67$, $F_2 \rightarrow (2/3)^2=0.44$, and $F_4 \rightarrow (2/3)^3=0.20$. It is easy to see this asymptotic behavior in the last graph, $T=10 T_c$, of Fig. 2. The first two graphs illustrate the importance of the relative observation time $T/T_c$. Indeed, for the short observation time, $T=0.1 T_c$, on the first graph the fidelities $F_1$, $F_2$, and $F_4$ never reach their optimum values even for large ratios of $D/l_c$. One can also remark that for small values of $D/l_c$ the fidelities $F_1$, $F_2$, and $F_4$ tend to their classical limits, $F_1 \rightarrow 1/2$, $F_2 \rightarrow 1/4$, and $F_4 \rightarrow 1/16$.
APPENDIX: PROPERTIES OF SPATIALLY MULTIMODE FREQUENCY NONDEGENERATE SQUEEZING

Let us consider first the squeezing transformation for the field amplitudes $\hat{S}(\vec{q},\vec{p})$ and $\hat{E}(\vec{q},\vec{p})$, defined with respect to the central frequency $\omega_0=\omega_1/2$ of the parametric down-conversion for the type-I collinear phase matching [23],

$$\hat{s}(\vec{q},\vec{\Omega}) = U(\vec{q},\vec{\Omega}) \hat{s}(\vec{q},\vec{\Omega}) + V(\vec{q},\vec{\Omega}) \hat{s}(-\vec{q},-\vec{\Omega}) .$$  

(A1)

Here $\vec{\Omega}$ is the frequency detuning from $\omega_0$. The coefficients $U(\vec{q},\vec{\Omega})$ and $V(\vec{q},\vec{\Omega})$ are given by

$$U(\vec{q},\vec{\Omega}) = \exp\{i[l(k_1 \vec{q},\vec{\Omega}) - k_2(-\vec{q},-\vec{\Omega})/2]\} \times \left( \cosh \Gamma(\vec{q},\vec{\Omega}) + \frac{i \delta(\vec{q},\vec{\Omega})}{2 \Gamma(\vec{q},\vec{\Omega})} \sinh \Gamma(\vec{q},\vec{\Omega}) \right) ,$$

$$V(\vec{q},\vec{\Omega}) = \exp\{i[l(k_1 \vec{q},\vec{\Omega}) - k_2(-\vec{q},-\vec{\Omega})/2]\} \times \frac{g}{\Gamma(\vec{q},\vec{\Omega})} \sinh \Gamma(\vec{q},\vec{\Omega}) .$$  

(A2)

Here $l$ is the length of the nonlinear crystal, and $k_{\pm}(\vec{q},\vec{\Omega})$ is the longitudinal component of the wave vector $\vec{k}(\vec{q},\vec{\Omega})$ for the wave with frequency $\omega_0+\vec{\Omega}$ and transverse component $\vec{q}$. The dimensionless mismatch function $\delta(\vec{q},\vec{\Omega})$ is given by

$$\delta(\vec{q},\vec{\Omega}) = [k_1(\vec{q},\vec{\Omega}) - k_2(-\vec{q},-\vec{\Omega}) - k_3]/l ,$$  

(A3)

where $k_3$ is the wave number of the pump wave. The central frequencies $\omega_1=\omega_0-\Omega_0$ and $\omega_2=\omega_0+\Omega_0$ of two bright beams of the frequency nondegenerate parametric down-conversion are found from the condition of zero mismatch, $\delta(q,\vec{\Omega})=\Omega_0=0$. We have assumed the paraxial approximation. The parameter $\Gamma(\vec{q},\vec{\Omega})$ is defined as

$$\Gamma(\vec{q},\vec{\Omega}) = \sqrt{g^2 - \delta^2(\vec{q},\vec{\Omega})}/4 ,$$  

(A4)

where $g$ is the dimensionless coupling strength of nonlinear interaction, taken as real for simplicity. It is proportional to the nonlinear susceptibility, the length of the crystal, and the amplitude of the pump field.

Consider now the slow amplitudes $\hat{S}_\mu(\vec{p},t)$, $\mu=1,2$, which are defined with respect to the frequencies $\omega_\mu$. In the Fourier domain we now have the frequency detuning $\Omega$, where $|\Omega|<\Omega_0$, from the central frequency of the corresponding beam: $\vec{\Omega}=-\omega_0+\Omega$ for the $\omega_1$ wave and $\vec{\Omega}=\omega_0+\Omega$ for the $\omega_2$ wave, so that $\hat{s}(\vec{q},\vec{\Omega})=-\omega_0+\Omega)\to s_1(\vec{q},\vec{\Omega})$, and $\hat{s}(\vec{q},\vec{\Omega})=\omega_0+\Omega)\to s_2(\vec{q},\vec{\Omega})$. The squeezing coefficients are rewritten in the form

$$U(\vec{q},-\Omega_0+\Omega) = U_1(\vec{q},\vec{\Omega}),$$

$$U(\vec{q},\Omega_0+\Omega) = U_2(\vec{q},\vec{\Omega}),$$

$$V(\vec{q},-\Omega_0+\Omega) = V_1(\vec{q},\vec{\Omega}),$$

$$V(\vec{q},\Omega_0+\Omega) = V_2(\vec{q},\vec{\Omega}).$$  

(A5)

Taking into account Eq. (A1) and the definitions above, we arrive at the squeezing equation (2.1).

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